# Supplemental Material: All sets of incompatible measurements give an advantage in quantum state discrimination

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## S1. INCOMPATIBILITY ROBUSTNESS – PRIMAL SDP FORMULATION

In this section we show the equivalence between (6) and the primal form of the SDP optimization problem (10) from the main text. The first constraint can be used to solve for the elements of the 'noise' POVM, namely

$$N_{a|x} = \frac{(1+r)\sum_\lambda p(a|x,\lambda)G_\lambda - M_{a|x}}{r} \qquad \forall a, x. \tag{S1}$$

By denoting $s = 1 + r$, the positivity of the POVM elements $N_{a|x}$ is then equivalent to

$$s\sum_\lambda p(a|x,\lambda)G_\lambda \geq M_{a|x} \qquad \forall a, x \tag{S2}$$

Now note that without loss of generality one can decompose the probabilities $p(a|x,\lambda)$ as a sum of deterministic probabilities, $p(a|x,\lambda) = \sum_{\mathbf{a}} D_{\mathbf{a}}(a|x)p(\mathbf{a}|\lambda)$, where $\mathbf{a} = \mathrm{a}_1\mathrm{a}_2\cdots\mathrm{a}_n$ is a string of outcomes (one for each value of x) and $D_{\mathbf{a}}(a|x) = \delta_{a,\mathbf{a}_x}$, i.e. such that $a = \mathrm{a}_x$ with certainty. We can then write

$$\sum_\lambda p(a|x,\lambda)G_\lambda = \sum_{\mathbf{a}} D_{\mathbf{a}}(a|x)G_{\mathbf{a}} \tag{S3}$$

where $G_{\mathbf{a}} = \sum_\lambda p(a|x,\lambda)G_\lambda$. Each $G_{\mathbf{a}}$ is positive semidefinite, and they sum to the identity operator, hence they form a valid POVM. This form of parent can be thought of as a canonical parent POVM. Finally, we note that we can define $\tilde{G}_{\mathbf{a}} = sG_{\mathbf{a}}$, which is a super-normalised POVM, i.e. such that

$$\tilde{G}_{\mathbf{a}} \geq 0 \quad \forall \mathbf{a} \tag{S4}$$

and

$$\sum_{\mathbf{a}} \tilde{G}_{\mathbf{a}} = s\sum_{\mathbf{a}} G_{\mathbf{a}} = s\mathbb{1}. \tag{S5}$$

Gathering the constraints (S2), (S4) and (S5), one obtains the primal SDP form

$$\begin{aligned} 1 + I_R(\{\mathbb{M}_x\}) = \min_{s,\{\tilde{G}_{\mathbf{a}}\}} \quad & s \\ \text{s.t.} \quad & \sum_{\mathbf{a}} D_{\mathbf{a}}(a|x)\tilde{G}_{\mathbf{a}} \geq M_{a|x} \\ & \sum_{\mathbf{a}} \tilde{G}_{\mathbf{a}} = s\mathbb{1}, \quad \tilde{G}_{\mathbf{a}} \geq 0 \end{aligned} \tag{S6}$$

We see that this is now explicitly in the form of an SDP, since all constraints are linear equalities or inequalities (given that $D_{\mathbf{a}}(a|x)$ are not variables, but are fixed functions).

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## S2. INCOMPATIBILITY ROBUSTNESS – DUAL FORMULATION

In this section we derive the dual SDP formulation of the RoI. The Lagrangian associated to the primal form of the SDP (S6) is given by

$$\mathcal{L} = s + \sum_{a,x} \mathrm{tr}\left[\omega_{ax}\left(M_{a|x} - \sum_{\mathbf{a}} D_{\mathbf{a}}(a|x)\tilde{G}_{\mathbf{a}}\right)\right] - \mathrm{tr}\left[X\left(s\mathbb{1} - \sum_{\mathbf{a}} \tilde{G}_{\mathbf{a}}\right)\right] - \mathrm{tr}\sum_{\mathbf{a}} y_{\mathbf{a}}\tilde{G}_{\mathbf{a}},$$

where we have introduced dual variables $\omega_{ax}$ and $y_{\mathbf{a}}$, which are taken to be positive-semidefinite for all $a$, $x$ and $\mathbf{a}$ respectively, and $X$ is an unrestricted dual variable. The constraints on the dual variables are imposed to ensure that the Lagrangian lower bounds the primal objective function whenever the primal constraints are satisfied. By grouping terms, the Lagrangian can be re-expressed as

$$\mathcal{L} = s(1 - \mathrm{tr}\, X) + \mathrm{tr}\sum_{a,x} \omega_{ax} M_{a|x} + \mathrm{tr}\sum_{\mathbf{a}} \tilde{G}_{\mathbf{a}}\left[X - \sum_{a,x} \omega_{ax} D_{\mathbf{a}}(a|x) - y_{\mathbf{a}}\right]$$

The Lagrangian becomes independent of the primal variables if we restrict to dual variables that satisfy $\mathrm{tr}\, X = 1$ and $X = \sum_{a,x} \omega_{ax} D_{\mathbf{a}}(a|x) + y_{\mathbf{a}}$ for all $\mathbf{a}$. In this case the Lagrangian becomes equal to $\mathrm{tr}\sum_{a,x} \omega_{ax} M_{a|x}$. Hence, the dual form of the SDP reads

$$\begin{aligned} 1 + I_R(\{\mathbb{M}_x\}) = \max_{\{\omega_{ax}\},X} \quad & \mathrm{tr}\sum_{a,x} \omega_{ax} M_{a|x} \\ \text{s.t.} \quad & X \geq \sum_{a,x} \omega_{ax} D_{\mathbf{a}}(a|x), \\ & \omega_{ax} \geq 0, \quad \mathrm{tr}\, X = 1 \end{aligned} \tag{S7}$$

The optimal values of the primal and the dual formulation coincide if strong duality holds. This is true if there exist a strictly feasible solution of the dual problem (and both problems are finite). An explicit strictly feasible solution is $X = \mathbb{1}/d$, $\omega_{ax} = \alpha\mathbb{1}$ for any $d$ and $\alpha$ such that $1/nd > \alpha > 0$. The existence of a strictly feasible solution thus ensures the equivalence between the primal and dual SDP formulations.

## S3. UPPER BOUND ON THE ADVANTAGE IN QSD FROM THE PRIMAL SDP

In this section we show that the RoI for a set of measurements upper bounds the advantage that the set of measurements has in the QSD game defined in the main text, compared to the optimal success which can be achieved with a single measurement. To see this, we start from the original formulation of the RoI (Eq. (6) from the main text). Let us denote by $G^*_\lambda$ and $p^*(a|x,\lambda)$ the optimal parent POVM attaining the minimum. Since the POVM elements of the noise $N_{a|x}$ are positive semi-definite, it follows that

$$[1 + I_R(\{\mathbb{M}_x\})] \sum_\lambda p^*(a|x,\lambda) G^*_\lambda \geq M_{a|x} \quad \forall a, x. \tag{S8}$$

By taking the trace on both sides with $\rho_{b|y}$, and by multiplying by the appropriate probabilities and summing, this implies that

$$\begin{aligned} &[1 + I_R(\{\mathbb{M}_x\})] \sum_{\substack{\lambda g\mu \\ abxy}} q(b,y)p(\mu)\,\mathrm{tr}\left[\rho_{b|y} G^*_\lambda\right] p^*(a|x,\lambda)p(x|y,\mu)p(g|a,y,\mu)\delta_{b,g} \\ &\qquad\qquad \geq \sum_{\mu abxyg} q(b,y)p(\mu)\,\mathrm{tr}\left[\rho_{b|y} M_{a|x}\right] p(x|y,\mu)p(g|a,y,\mu)\delta_{b,g}, \end{aligned} \tag{S9}$$

where $\mathcal{E}_y = \{q(b|y), \rho_{b|y}\}_b$ represents the ensembles for a QSD game, which occur with probability $q(y)$ (such that $q(y)q(b|y) = q(b,y)$) and $p(g|a,y,\lambda)$ the guessing strategy, as given in (2) from the main text. Let us define

$$p(g|\lambda,y,\mu) = \sum_{a,x} p^*(a|x,\lambda)p(x|y,\mu)p(g|a,y,\mu) \tag{S10}$$

so that (S9) reads

$$[1+I_R(\{\mathbb{M}_x\})]\sum_{\substack{\lambda g\mu\\ by}} q(b,y)p(\mu)\,\mathrm{tr}\left[\rho_{b|y}G^*_\lambda\right]p(g|\lambda,y,\mu)\delta_{b,g}$$

$$\geq \sum_{\mu abxyg} q(b,y)p(\mu)\,\mathrm{tr}\left[\rho_{b|y}M_{a|x}\right]p(x|y,\mu)p(g|a,y,\mu)\delta_{b,g}, \quad \text{(S11)}$$

The sum on the left hand side has the form of the success probability in QSD game with a single measurement, given in (3). It does not have the most general form, since $G^*_\lambda$ does not depend on $\mu$ (in this expression, $\lambda$ is playing the role of $a$ in (3)). Hence, the sum is not larger than the optimal sucess probability with single measurement in the QSD game:

$$[1+I_R(\{\mathbb{M}_x\})]P_g^{\mathrm{C}}(\{\mathcal{E}_y\}) \geq \sum_{\substack{\mu,a,b\\ x,y,g}} q(b,y)p(\mu)p(x|y,\mu)\,\mathrm{tr}\left[\rho_{b|y}M_{a|x}\right]p(g|a,y,\mu)\delta_{b,g}. \quad \text{(S12)}$$

This expression holds for all $p(\mu)$, $p(x|y,\mu)$ and $p(g|a,y,\mu)$, so it must hold if we maximise both sides over all such probabilities (noting that the left hand side is in fact already independent of all of them):

$$[1+I_R(\{\mathbb{M}_x\})]P_g^{\mathrm{C}}(\{\mathcal{E}_y\}) \geq \max_{\substack{p(\mu)\\ p(x|y,\mu)\\ p(g|a,y,\mu)}} \sum_{\mu abxyg} q(b,y)p(\mu)p(x|y,\mu)\,\mathrm{tr}\left[\rho_{b|y}M_{a|x}\right]p(g|a,y,\mu)\delta_{b,g}. \quad \text{(S13)}$$

The right-hand-side is now equal to the optimal success in the QSD game with incompatible measurements as defined in (2). This holds for all QSD games, (collections of ensembles $\{\mathcal{E}_y\}_y$. Thus, re-arranging and maximising over all games we arrive at the following inequality

$$1+I_R(\{\mathbb{M}_x\}) \geq \max_{\{\mathcal{E}_y\}} \frac{P_g(\{\mathcal{E}_y\},\{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{E}_y\})}. \quad \text{(S14)}$$

This proves that upper bound, that $1+I_R(\{\mathbb{M}_x\})$ is always larger than the advantage in any QSD game.

## S4. LOWER BOUND

In this section we now show that the upper bound from the previous section can be achieved, by exhibiting a carefully chosen optimal game $\{\mathcal{E}^*_y\}_y$, that has advantage equal to $1+I_R(\{\mathbb{M}_x\})$ when played with $\{\mathbb{M}_x\}_x$.

Consider the optimal dual variables $\omega^*_{ax}$ and $X^*$ from the dual SDP formulation of the RoI as defined in (S7). Those variables satisfy

$$\begin{aligned} 1+I_R(\{\mathbb{M}_x\}) &= \mathrm{tr}\sum_{a,x}\omega^*_{ax}M_{a|x}, \\ \mathrm{tr}[X^*] = 1&, \qquad \omega^*_{ax}\geq 0 \\ X^* &\geq \sum_{a,x} D_{\mathbf{a}}(a|x)\omega^*_{a,x}, \qquad \forall \mathbf{a}. \end{aligned} \quad \text{(S15)}$$

Let us now introduce the following auxiliary variables

$$N^* = \mathrm{tr}\sum_{a,x}\omega^*_{ax}, \qquad q^*(a,x) = \frac{\mathrm{tr}\,\omega^*_{ax}}{N^*}, \qquad \rho^*_{a|x} = \frac{\omega^*_{ax}}{\mathrm{tr}\,\omega^*_{ax}} = \frac{\omega^*_{ax}}{N^*q^*(a,x)}.$$

The variables $\rho^*_{a|x}$ are normalised quantum states for all $a$, $x$ by construction, while $\{q^*(a,x)\}$ is a normalised probability distribution. By using the auxiliary variables the first constraint from (S15) reduces to

$$1+I_R(\{\mathbb{M}_x\}) = N^*\sum_{a,x} q^*(a,x)\,\mathrm{tr}\left[\rho^*_{a|x}M_{a|x}\right] \quad \text{(S16)}$$

Let us now assume that the QSD game is played with the set of ensembles $\{\mathcal{E}^*_y\}_y$, where $\mathcal{E}^*_y = \{q^*(b|y),\rho^*_{b|y}\}$, $q^*(b|y) = q^*(b,y)/q^*(y)$, and $q^*(y) = \sum_b q^*(b,y)$ is the probability that Bob sends $y$ to Alice. The strategy for playing the game is taken to be the following:

- $p(\mu) = \delta_{\mu,0}$,
- $p(x|y, \mu = 0) = \delta_{y,x}$, *i.e.* we measure $\mathbb{M}_y$ when given $y$,
- $p(g|a, y, \mu = 0) = \delta_{g,b}$, *i.e.* we guess that $b = g = a$ when get outcome $a$.

The score achieved by this strategy is a lower bound on $P_g(\{\mathcal{E}_y^*\}, \{\mathbb{M}_x\})$, (since this is a potentially sub-optimal strategy for playing). It therefore holds that

$$\begin{aligned} P_g(\{\mathcal{E}_y^*\}, \{\mathbb{M}_x\}) &\geq \sum_{\substack{a,b,x \\ y,g,\mu}} q^*(b, y)\delta_{\mu,0}\delta_{x,y} \operatorname{tr}\left[\rho^*_{b,y} M_{a|x}\right] \delta_{g,b}\delta_{a,g} \\ &= \sum_{a,x} q^*(a, x) \operatorname{tr}\left[\rho^*_{a|x} M_{a|x}\right] \\ &= \frac{1}{N^*}(1 + I_R(\{\mathbb{M}_x\})) \end{aligned} \tag{S17}$$

As a short digression, which will be useful later, let us look more carefully at the strategies $P_g^{\mathrm{C}}(\{\mathcal{E}_y\})$:

$$P_g^{\mathrm{C}}(\{\mathcal{E}_y\}) = \max_{\substack{\mathbb{G}_\nu \\ p(g|y,\nu) \\ p(\nu)}} \sum_{\substack{a,b,y \\ g,\nu}} q(b, y)p(\nu) \operatorname{tr}\left[\rho_{b,y} G_{a|\nu}\right] p(g|a, y, \nu)\delta_{b,g} \tag{S18}$$

In in the first section of the appendix, one can decompose $p(g|a, y, \nu)$ into deterministic distributions. For that purpose introduce $D_{\mathbf{b}}(g|y) = \delta_{g,\mathrm{b}_x}$ to be functions such that $g$ is deterministically equal to $b_x$ where $\mathbf{b}$ is a string of outcomes, one for each measurement setting. It is always possible to write

$$p(g|a, y, \nu) = \sum_{\mathbf{b}} p(\mathbf{b}|a, \nu)D_{\mathbf{b}}(g|y) \tag{S19}$$

This decomposition allows one to obtain

$$\begin{aligned} \sum_{\substack{a,b,y \\ g,\nu}} q(b, y)p(\nu) \operatorname{tr}\left[\rho_{b|y} G_{a|\nu}\right] p(g|a, y, \nu)\delta_{b,g} &= \sum_{b,y,\mathbf{b}} q(b, y) \operatorname{tr}\left[\rho_{b|y} \left(\sum_{a,\nu} p(\nu)G_{a|\nu} p(\mathbf{b}|a, \nu)\right)\right] D_{\mathbf{b}}(g|y)\delta_{b,g} \\ &= \sum_{b,y,\mathbf{b}} q(b, y) \operatorname{tr}\left[\rho_{b,y} \tilde{G}_{\mathbf{b}}\right] D_{\mathbf{b}}(g|y)\delta_{b,g} \end{aligned} \tag{S20}$$

where to obtain the third line we introduced the new variable

$$\tilde{G}_{\mathbf{b}} = \sum_{a,\nu} p(\nu)G_{a|\nu} p(\mathbf{b}|a, \nu) \tag{S21}$$

For all values of $\mathbf{b}$ this variable is positive semi-definite and it satisfies the following completeness relation

$$\begin{aligned} \sum_{\mathbf{b}} \tilde{G}_{\mathbf{b}} &= \sum_{a,\mathbf{b},\nu} p(\nu)G_{a|\nu} p(\mathbf{b}|a, \nu) \\ &= \sum_{a,\nu} p(\nu)G_{a|\nu} \\ &= \sum_{\nu} p(\nu)\mathbb{1} \\ &= \mathbb{1} \end{aligned} \tag{S22}$$

The second equality is a simple consequence of the fact that $p(\mathbf{b}|a, \nu)$ is a probability distribution, while the third one comes from the fact that $G_{a|\nu}$ is a valid measurement. Hence, positivity and completeness of $\tilde{G}_{\mathbf{b}}$ ensure that it represents a valid POVM. Eq. (S20) means that we can, without loss of generality, assume that we measure $\rho_{b|y}$ in order to make a guess for every possible value of $b$ for each $y$ and later simply announce the value $g = b_y$ once we know $y$. The above shows that this is in fact as good as the most general strategy and thus

$$P_g^{\mathrm{C}}(\{\mathcal{E}_y\}) := \max_{\tilde{\mathbb{G}}} \sum_{b,y,\mathbf{b}} q(b, y) \operatorname{tr}\left[\rho_{b,y} \tilde{G}_{\mathbf{b}}\right] D_{\mathbf{b}}(g|y)\delta_{b,g} \tag{S23}$$

Let us now return to the variable $N^*$. From the definition of $N^*$ and the dual SDP formulation it follows

$$X^* \geq \sum_{b,y} D_{\mathbf{b}}(b|y) N^* q^*(b,y) \rho^*_{b|y}.$$

Multiplying by and arbitrary $\tilde{G}_{\mathbf{b}}$, summing over $\mathbf{b}$ and tracing leads to

$$\operatorname{tr} \sum_{\mathbf{b}} X^* \tilde{G}_{\mathbf{b}} \geq \sum_{b,y,\mathbf{b}} D_{\mathbf{b}}(g|y) \delta_{b,g} N^* q^*(b,y) \operatorname{tr}\left[\tilde{G}_{\mathbf{b}} \rho^*_{b,y}\right] \tag{S24}$$

Since $\tilde{G}_{\mathbf{b}}$ is a valid POVM and $X^*$ has unit trace the left-hand-side of the inequality is equal to one. As it holds for all $\tilde{G}_{\mathbf{b}}$, it holds if the expression is maximized over $\tilde{G}_{\mathbf{b}}$, which implies

$$\max_{\tilde{G}_{\mathbf{b}}} \frac{1}{N^*} \geq \max_{\tilde{G}_{\mathbf{b}}} \sum_{b,y,\mathbf{b}} q^*(b,y) \operatorname{tr}\left[\tilde{G}_{\mathbf{b}} \rho^*_{b,y}\right] D_{\mathbf{b}}(g|y) \delta_{b,g}.$$

This furthermore implies

$$\frac{1}{N^*} \geq P_g^{\mathrm{C}}(\{\mathcal{E}^*_y\}). \tag{S25}$$

This inequality, together with (S17) implies

$$\frac{P_g(\{\mathcal{E}^*_y\}, \{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{E}^*_y\}} \geq 1 + I_R(\{\mathbb{M}_x\}). \tag{S26}$$

However, since we already proved in (S14) that $1 + I_R(\{\mathbb{M}_x\})$ upper bounds the success probability for any QSD game $\{\mathcal{E}_y\}_y$, it must be the case that $\{\mathcal{E}^*_y\}_y$ is equal to $1 + I_R(\{\mathbb{M}_x\})$, which completes the proof of the main result.

## S5. MONOTONES FOR MEASUREMENT SIMULATION

In this section we prove that the measurements $\{\mathbb{M}_x\}_x$ can simulate another set of measurements $\{\mathbb{M}'_y\}_y$ if and only if $\{\mathbb{M}'_y\}_y$ never outperforms $\{\mathbb{M}_x\}_x$ in the QSD game introduced in the main text for every ensemble of states:

$$\{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\} \iff P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\}) \geq P_g(\{\mathcal{E}_z\}, \{\mathbb{M}'_y\}) \quad \forall \{\mathcal{E}_z\}. \tag{S27}$$

Recall that the success in the QSD game is defined as (we change notation here slightly, using $z$ and $c$ for the QSD game, as we will use $b$ and $y$ for the measurements $\{\mathbb{M}'_y\}$):

$$P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\}) = \max_{\substack{p(x|z,\mu) \\ p(g|a,z,\mu) \\ p(\mu)}} \sum_{\substack{a,c,g \\ x,z,\mu}} q(c,z) p(\mu) p(x|z,\mu) \operatorname{tr}\left[\rho_{c|z} M_{a|x}\right] p(g|a,z,\mu) \delta_{c,g}. \tag{S28}$$

By introducing a new set of measurements $\{\mathbb{M}'_z\}_z$, where $\mathbb{M}'_z = \{M_{g|z}\}_g$, which can be simulated by $\{\mathbb{M}_x\}_x$ according to the definition of the simulation

$$M'_{g|z} = \sum_{a,x,\mu} p(\mu) p(x|y,\mu) M_{a|x} p(g|a,z,\mu) \quad \forall b, y \tag{S29}$$

the success probability $P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\})$ can be re-expressed in a conceptually simpler form:

$$P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\}) = \max_{\{\mathbb{M}'_z\} \prec \{\mathbb{M}_x\}} \sum_{c,z,g} q(c,z) \operatorname{tr}\left[\rho_{c|z} M'_{g|z}\right] \delta_{c,g} \tag{S30}$$

That is, we see that the optimisation carried out can be thought of as optimising over all measurements $\{\mathbb{M}'_z\}_z$ that can be simulated by $\{\mathbb{M}_x\}_x$, where by definition now the outcome of the measurement is the guess $g$ of the corresponding state $c$ from the ensemble.

Given this equivalent formulation, it is immediate that one direction of (S27) is immediately satisfied:

$$\{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\} \implies P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\}) \geq P_g(\{\mathcal{E}_z\}, \{\mathbb{M}'_y\}) \quad \forall \{\mathcal{E}_z\}. \tag{S31}$$

Now we want to prove the converse direction. For that purpose assume $P_g(\{\mathcal{E}_z\}, \{\mathbb{M}_x\}) - P_g(\{\mathcal{E}_z\}, \{\mathbb{M}'_y\}) \geq 0$ for all QSD games $\{\mathcal{E}_z\}_z$. This assumption, written in full is

$$\max_{\substack{p(x|z,\mu)\\p(g|a,z,\mu)\\p(\mu)}} \sum_{\substack{a,c,g\\x,z,\mu}} q(c,z)p(\mu)p(x|z,\mu)\,\mathrm{tr}\left[\rho_{c|z}M_{a|x}\right]p(g|a,z,\mu)\delta_{b,g}$$

$$-\max_{\substack{p'(y|z,\nu)\\p'(g|b,z,\nu)\\p'(\nu)}} \sum_{\substack{b,c,g\\y,z,\nu}} q(c,z)p'(\nu)p'(y|z,\nu)\,\mathrm{tr}\left[\rho_{c|z}M'_{b|y}\right]p'(g|b,z,\nu)\delta_{c,g} \geq 0 \quad \text{(S32)}$$

Let us now make a guess for a possibly sub-optimal strategy:

- $p'(nu) = \delta_{\nu,0}$,
- $p'(y|z,\nu=0) = \delta_{y,z}$,
- $p'(g|b,z,\nu=0) = \delta_{g,b}$.

This strategy implies

$$\max_{\substack{p(x|z,\mu)\\p(c|a,z,\mu)\\p(\mu)}} \sum_{\substack{a,c\\x,z,\mu}} q(c,z)p(\mu)p(x|z,\mu)\,\mathrm{tr}\left[\rho_{c|z}M_{a|x}\right]p(c|a,z,\mu) - \sum_{c,z} q(c,z)\,\mathrm{tr}\left[\rho_{c|z}M'_{c|z}\right] \geq 0$$

which after re-arranging gives

$$\max_{\substack{p(x|z,\mu)\\p(c|a,z,\mu)\\p(\mu)}} \sum_{c,z} q(c,z)\,\mathrm{tr}\left[\rho_{c|z}\left(\sum_{a,x,\mu} p(\mu)p(x|z,\mu)M_{a|x}p(c|a,z,\mu) - M'_{c|z}\right)\right] \geq 0 \quad \text{(S33)}$$

This must be true for all $\{\mathcal{E}_z\}_z$, with $\mathcal{E}_z = \{q(c|z), \rho_{c|z}\}_c$. It therefore holds if minimised over all such QSD games:

$$\min_{\{\mathcal{E}_z\}} \max_{\substack{p(x|z,\mu)\\p(c|a,z,\mu)\\p(\mu)}} \sum_{c,z} q(c,z)\,\mathrm{tr}\left[\rho_{c|z}\left(\sum_{a,x,\mu} p(\mu)p(x|z,\mu)M_{a|x}p(c|a,z,\mu) - M'_{c|z}\right)\right] \geq 0 \quad \text{(S34)}$$

This expression is linear in $\{\mathcal{E}_z\}_z$, *i.e.* in $\sigma_{c|z} = q(c,z)\rho_{c|z}$, which means also convex in these variables, and it is concave in $\{p(x|z,\mu), p(c|a,z,\mu), p(\mu)\}$. Therefore we can apply the minimax theorem [1] and interchange the minimization and maximization. The last inequality, thus, reads

$$\max_{\substack{p(x|z,\mu)\\p(c|a,z,\mu)\\p(\mu)}} \min_{\{\mathcal{E}_z\}} \sum_{c,z} q(c,z)\,\mathrm{tr}\left[\rho_{c|z}\Delta_{cz}\right] \geq 0, \quad \text{(S35)}$$

where we have introduced

$$\Delta_{cz} = \sum_{a,x,\mu} p(\mu)p(x|z,\mu)M_{a|x}p(c|a,z,\mu) - M'_{c|z} \quad \text{(S36)}$$

Now, If $\{\mathbb{M}_x\} \succ \{\mathbb{M}'_z\}$, there exist $p(x|z,\mu), p(c|a,z,\mu)$ and $p(\mu)$ such that $\Delta_{cz} = 0$ for all values of $c$ and $z$. Let us assume that this is not true – i.e. that no such $p(x|z,\mu), p(c|a,z,\mu)$ and $p(\mu)$ exist, in other words that $\Delta_{cz} \neq 0$ for all $c$ and $z$. In what follows we will show, by contradiction, that this is impossible.

First, note that

$$\begin{aligned}
\sum_c \Delta_{cz} &= \sum_{a,c,x,\mu} p(\mu)p(x|z,\mu)M_{a|x}p(c|a,z,\mu) - \sum_c M'_{c|z} \\
&= \sum_{a,x,\mu} p(\mu)p(x|z,\mu)M_{a|x} - \mathbb{1} \\
&= \sum_{x,\mu} p(\mu)p(x|z,\mu)\mathbb{1} - \mathbb{1} \qquad \text{(S37)} \\
&= 0 \qquad \text{(S38)}
\end{aligned}$$

The second line is a consequence of the normalisation of $p(c|a,z,\mu)$ and completeness of each $\mathbb{M}'_z$. The third line follows from the completeness of each $\mathbb{M}_x$ and the last from from the normalisation of $p(\mu)$ and $p(x|z,\mu)$. Since $\sum_c \Delta_{cz} = 0$ it is impossible that $\Delta_{cz} \geq 0$ for all $c, z$, since this would only happen if all $\Delta_{cz}$ vanished identically, but by assumption this isn't the case.

Hence, for each $z$, there must be at least one $c^*(z)$ such that $\Delta_{c^*(z)z}$ has a negative eigenvalue. Let us denote by $|\psi_{c^*(z)z}\rangle$ the corresponding eigenvector with eigenvalue $\psi_{c^*(z)z} < 0$. Now let us choose $\{\mathcal{E}^*_z\}_z$ such that

- $q^*(c,z) = q^*(c|z)q^*(z)$,
- $q^*(z) = 1/n$,
- $q^*(c|z) = \delta_{c^*(z),c}$,
- $\rho^*_{c^*(z)|z} = |\psi_{c^*(z)z}\rangle\langle\psi_{c^*(z)|z}|$

Then

$$\sum_{c,z} q^*(c,z) \operatorname{tr}\left[\rho^*_{c|z}\Delta_{cz}\right] = \frac{1}{n}\sum_z \psi_{c^*(z)z} < 0, \tag{S39}$$

which is a contradiction, since by assumption $\sum_{cz} q(c,z) \operatorname{tr}\left[\rho_{c|z}\Delta_{cz}\right] \geq 0$. Therefore, there must exist $p(x|z,\mu), p(c|a,z,\mu)$ and $p(\mu)$ such that $\sum_{a,x,\mu} p(\mu)p(x|z,\mu)M_{a|x}p(c|a,z,\mu) = M'_{c|z}$ and hence $\{\mathbb{M}_x\} \succ \{\mathbb{M}'_z\}$. By this we have proven that

$$P_g(\{\mathcal{E}_z\},\{\mathbb{M}_x\}) \geq P_g(\{\mathcal{E}_z\},\{\mathbb{M}'_y\}) \quad \forall\{\mathcal{E}_z\} \implies \{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\} \tag{S40}$$

which together with the already proven converse statements implies (S27). In words, this shows that the guessing probabilities for all QSD games $\{\mathcal{E}_z\}_z$ constitute a complete set of monotones for the partial order $\{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\}$.

Finally, let us show how this relates to the RoI. Assume $\{\mathbb{M}_x\}_x$ has optimal QSD game $\{\mathcal{E}^*_z\}_z$ such that $1 + I_R(\{\mathbb{M}_x\}) = P_g(\{\mathcal{E}^*_z\},\{\mathbb{M}_x\})/P_g^{\mathrm{C}}(\{\mathcal{E}^*_z\})$. Analogously, assume $\{\mathbb{M}'_y\}$ has the optimal game $\{\mathcal{F}^*_z\}_z$ such that $1 + I_R(\{\mathbb{M}'_y\}) = P_g(\{\mathcal{F}^*_z\},\{\mathbb{M}'_y\})/P_g^{\mathrm{C}}(\{\mathcal{F}^*_z\})$. Let us assume $\{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\}$. Then

$$\begin{aligned}
1 + I_R(\{\mathbb{M}_x\}) &= \frac{P_g(\{\mathcal{E}^*_z\},\{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{E}^*_z\})} \\
&\geq \frac{P_g(\{\mathcal{F}^*_z\},\{\mathbb{M}_x\})}{P_g^{\mathrm{C}}(\{\mathcal{F}^*_z\})} \\
&\geq \frac{P_g(\{\mathcal{F}^*_z\},\{\mathbb{M}'_y\})}{P_g^{\mathrm{C}}(\{\mathcal{F}^*_z\})} \\
&= 1 + I_R(\{\mathbb{M}'_y\})
\end{aligned}$$

The first inequality follows from the fact that $\{\mathcal{E}^*_z\}_z$ is the optimal QSD game for $\{\mathbb{M}_x\}_x$. The second inequality follows from from (S27). Thus we conclude that $I_R(\{\mathbb{M}_x\}) > I_R(\{\mathbb{M}'_y\})$ whenever $\{\mathbb{M}_x\} \succ \{\mathbb{M}'_y\}$, *i.e.* the RoI is also a monotone for measurement simulation.

[1] J. V. Neumann, Mathematische Annalen **100**, 295–320 (1928).